A generalized Weyl relation approach to the time operator and its connection to the survival probability [*]

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The time operator, an operator which satisfies the canonical commutation relation with the Hamiltonian, is investigated, on the basis of a certain algebraic relation for a pair of operators T and H, where T is symmetric and H self-adjoint. This relation is equivalent to the Weyl relation, in the case of self-adjoint T, and is satisfied by the Aharonov-Bohm time operator T_0 and the free Hamiltonian H_0 for the one-dimensional free-particle system. In order to see the qualitative properties of T_0, the operators T and H satisfying this algebraic relation are examined. In particular, it is shown that the standard deviation of T is directly connected to the survival probability, and H is absolutely continuous. Hence, it is concluded that the existence of the operator T implies the existence of scattering states. It is also shown that the minimum uncertainty states do not exist. Other examples of these operators T and H, than the one-dimensional free-particle system, are demonstrated.

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I. INTRODUCTION

The concept of the time operator is strongly connected with the time-energy uncertainty relation. The time operator, denoted by T, is usually defined to satisfy the canonical commutation relation (CCR) with the Hamiltonian H: $[T, H] = i$ (see [1] and the references therein). If such an operator were defined consistently on the Hilbert space corresponding to a certain quantum system, then the time-energy uncertainty relation could be automatically reduced from the Cauchy-Schwarz inequality, as in the case between the position and momentum operators on $L^2(\mathbf{R}^1)$. For instance, if we take the operator T_0 suggested by Aharonov and Bohm [2], as a time operator for the one-dimensional free-particle system (1DFPS), we formally have $[T_0, H_0] = i$ and derive the uncertainty relation between T_0 and H_0. Here $H_0 := P^2/2$ is the free Hamiltonian for the 1DFPS, and T_0 is defined as

$$T_0 := \frac{1}{2}\left(QP^{-1} + P^{-1}Q\right), \tag{1}$$

where Q and P are the position and momentum operators on $L^2(\mathbf{R}^1)$ (more precise definition is given in Sec. III). T_0 is often called the Aharonov-Bohm time operator. It is, however, not clear whether the inverse P^{-1} could be well-defined. We should also remember the criticism posed by Pauli [3], although it is not rigorous, that the time operator can not necessarily be defined for all quantum systems without contradiction. Furthermore the physical meaning of the time operator, if any, still remains unclear.

We shall base our discussion on the axiomatic quantum mechanics. Then it is possible to comment on the above difficulties from the axiomatic points of view. We first see that the inverse P^{-1} is a well-defined self-adjoint operator on $L^2(\mathbf{R}^1)$ (more details are given in Sec. III). Recently, the operator T_0 was shown to be well-defined, and its mathematical character

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was clarified, through the study of the time-of-arrival problem [4]. Observe that in Pauli's criticism, it is implicitly assumed that if there exists a self-adjoint operator T which satisfies the CCR with the Hamiltonian H for some system,

$$TH\psi - HT\psi = i\psi, \quad \forall \psi \in \mathrm{Dom}(TH) \cap \mathrm{Dom}(HT), \tag{2}$$

one would be able to derive the following relation

$$He^{i\epsilon T}\psi = e^{i\epsilon T}(H + \epsilon)\psi, \quad \forall \psi \in \mathrm{Dom}(H), \ \forall \epsilon \in \mathbf{R}^1. \tag{3}$$

We have to be careful, however, about this kind of logic, since it is not generally true whereas its converse is true. For example, consider a pair of operators, the position and momentum operators on $L^2([0,1])$, Q and P, for the above T and H, respectively [5]. They satisfy Eq. (2) but Eq. (3) is satisfied only for the particular values $\epsilon = 2\pi n$, $n \in \mathbf{Z}$, since, in order that P be self-adjoint, the domain $\mathrm{Dom}(P)$ has to be supplemented with a boundary condition $\psi(0) = \theta\psi(1)$ with a fixed $\theta \in \mathbf{C}$, $|\theta| = 1$, $\forall \psi \in \mathrm{Dom}(P)$ [6]. Furthermore, there is no a priori reason why we have to consider the time operator an observable, that is, a self-adjoint operator: we don't have any interpretation of the time operator as an observable. In this paper, we shall require the time operator be symmetric, satisfying Eq. (2) with the Hamiltonian, but not necessarily be self-adjoint.

The investigation of the time operator is important to understanding the time-energy uncertainty relation, and may have a significance for the analysis of the dynamics of quantum systems. A reason for the latter is that the time operator is directly connected to the Hamiltonian through the CCR, and this is algebraically so strong relation between operators as to prescribe qualitative aspects of their spectra, we can expect that the time operator brings us information about qualitative aspects of the time evolution of quantum systems. Hence, our purposes here are to examine for which quantum systems such a symmetric time operator is allowed to exist consistently, and to disclose its relevance to the dynamics of the quantum system under consideration.

From what is mentioned above, the investigation of the time operator is involved in that of the commutator (not necessarily canonical). The connection between the commutator of the form $[H, iA] = C$ ($C \geq 0$) and the spectra of self-adjoint operators, H, A and C, has been widely studied by Putnam [7], Kato [8], Lavine [9], [10], and others (see also [11]). We here, however, restrict our consideration to the more strong form, which will be called the "T-weak" Weyl relation.

Definition 1.1 : Let $\mathcal{H}$ be a Hilbert space, T be a symmetric operator on $\mathcal{H}$, and H be a self-adjoint operator on $\mathcal{H}$. If, for any $\psi \in \mathrm{Dom}(T)$ and for any $t \in \mathbf{R}^1$, the relations $e^{-itH}\psi \in \mathrm{Dom}(T)$ and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi \tag{4}$$

hold, then a pair of operators T and H is said to satisfy the T-weak Weyl relation (T-weak WR), or T (H) is said to satisfy the T-weak WR with H (T).

One can find, from the above definition, that $\mathrm{Dom}(Te^{-itH}) = \mathrm{Dom}(T)$, $\forall t \in \mathbf{R}^1$. Thus the T-weak WR are represented merely by

$$Te^{-itH} = e^{-itH}(T + t), \quad \forall t \in \mathbf{R}^1. \tag{5}$$

It will follow that the time operator T_0, in Eq. (1), is a symmetric operator on $L^2(\mathbf{R}^1)$ and one of its symmetric extensions, denoted by $\widetilde{T_0}$, satisfies the $\widetilde{T_0}$-weak WR with H_0 (see Sec. III). Thus as long as to see the qualitative properties of T_0 (or $\widetilde{T_0}$), it may suffice to examine the T-weak WR and the operators T and H satisfying this relation, by paying a particular attention to their spectra and to the uncertainty relation between them. We have obtained

the fact that the time operator is deeply connected to the survival probability. Indeed, if a pair of operators T and H satisfies the T-weak WR, the following inequality

$$\frac{4\,(\Delta T)_\psi^2 \,\|\psi\|^2}{t^2} \geq \left|\langle \psi, e^{-itH}\psi \rangle\right|^2 \tag{6}$$

holds for every $\psi \in \mathrm{Dom}(T)$ and for every $t \in \mathbf{R}^1 \backslash \{0\}$, where $(\Delta T)_\psi$ is the standard deviation of T with respect to ψ, and $\left|\langle \psi, e^{-itH}\psi \rangle\right|^2$ is the survival probability of ψ at time t. This is shown in Theorem 4.1. As an application of this inequality, we have Corollary 4.3 which states that H has no point spectrum. Furthermore, it is shown that H is absolutely continuous [12], as in Theorem 4.4. This means that the existence of the time operator, which satisfies the T-weak WR with the Hamiltonian for some system, infers that the system consists of only scattering states. Also, in Theorem 5.1, the absence of minimum-uncertainty states, for the uncertainty relation between T and H, is proved, under some condition satisfied by the operators $\widetilde{T_0}$ and H_0.

In Sec. II, the connection among the CCR, Weyl relation, and T-weak WR is mentioned. Section III is devoted to the brief study of the Aharonov-Bohm time operator in Eq. (1), to see a sign of the deep connection between the operator T and the survival probability, followed by several statements in Sec. IV. They include the inequality (6) and the spectral properties of both T and H, e.g. Theorem 4.4. Theorem 5.1 is proved in Sec. V. Further discussion about the time operator is developed in Sec. VI, on the basis of the results of the preceding sections and of the theory of Schrödinger operators. We mention other quantum systems than the 1DFPS for which an operator T exists, to satisfy the T-weak WR with the Hamiltonian. In fact, for a certain class of quantum systems, time operators are easily constructed by unitary transformations of $\widetilde{T_0}$. Concluding remarks are given in Sec. VII.

II. THE CANONICAL COMMUTATION RELATION, WEYL RELATION, AND T-WEAK WEYL RELATION

The T-weak WR in Eq. (4) or (5) is characterized more clearly, in the Heisenberg picture. The T-weak WR is represented, in an alternative form, as

$$T_t = T + tI, \quad \forall t \in \mathbf{R}^1, \tag{7}$$

where $T_t := e^{itH} T e^{-itH}$. It is now clear that T, which satisfies the T-weak WR with H, is shifted proportionally to the time parameter t in the Heisenberg picture. This fact bring us an image of time for T. We also see, from this form, that T is necessarily unbounded. It is, however, noted that in our investigation the T-weak WR in Eq. (5) is more convenient than in Eq. (7). The connection among the Weyl relation (WR) [13], the CCR and the T-weak WR is very important, when one considers whether a symmetric operator T, satisfying the T-weak WR with the Hamiltonian for some system, is the time operator. Recall that the latter is defined as a symmetric operator satisfying the CCR with the same Hamiltonian as in Eq. (2). In this respect, we put forward the next proposition.

*Proposition **2.1** : Let $\mathcal{H}$ be a Hilbert space, T be a closed symmetric operator on $\mathcal{H}$, and H be a self-adjoint operator on $\mathcal{H}$. If a pair of operators T and H satisfies the T-weak WR, then there is a dense subspace $\mathcal{D} \subset \mathcal{H}$ such that*

(i) $\mathcal{D} \subset \mathrm{Dom}(TH) \cap \mathrm{Dom}(HT)$,
(ii) $H : \mathcal{D} \to \mathcal{D}$,
(iii) The CCR holds in the meaning of that $TH - HT = i$ on $\mathrm{Dom}(TH) \cap \mathrm{Dom}(HT)$.

Moreover, if T is self-adjoint, then the operators T and H satisfy the WR,

$$e^{-isT} e^{-itH} = e^{-ist} e^{-itH} e^{-isT}, \quad \forall s, \ \forall t \in \mathbf{R}^1. \tag{8}$$

The above (i), (ii), and (iii) are proved in the same manner as in the proof [14], by noting the strong continuity of $Te^{-itH}\psi$, $\forall\psi \in \text{Dom}(T)$, by virtue of the T-weak WR, and the closedness of T, and also by considering the subspace spaned by the following subset of $\mathcal{H}$, as a subspace $\mathcal{D}$ in this proposition,

$$\left\{ \psi_f \in \mathcal{H} \ \middle| \ \psi_f := \int_{-\infty}^{\infty} f(s)e^{-isH}\psi ds, \ \forall f \in C_0^{\infty}(\mathbf{R}^1) \text{ and } \forall\psi \in \text{Dom}(T) \right\},$$

where the integral is defined by Riemann's sense and thus a strong limit. The last part of the proposition is proved as follows. In the case of T being self-adjoint, we see, from the T-weak WR (5), that $\forall\phi \in \mathcal{H}$ and $\forall\psi \in \text{Dom}(T)$,

$$\int_{\mathbf{R}^1} \lambda d \left\langle \phi, e^{itH}F(\lambda)e^{-itH}\psi \right\rangle = \left\langle \phi, e^{itH}Te^{-itH}\psi \right\rangle = \left\langle \phi, (T+t)\psi \right\rangle$$

$$= \int_{\mathbf{R}^1} (\lambda + t)d \left\langle \phi, F(\lambda)\psi \right\rangle = \int_{\mathbf{R}^1} \lambda d \left\langle \phi, F_t(\lambda)\psi \right\rangle,$$

where $\{F(B) \mid B \in \mathbf{B}^1\}$ is the spectral measure of T, $\mathbf{B}^1$ is the σ-field which is generated by all open sets of $\mathbf{R}^1$, and $F_t(B) := F(\{\lambda - t \mid \lambda \in B\})$. From the uniqueness of the spectral resolution, this means that $e^{itH}F(B)e^{-itH} = F_t(B)$, for all $t \in \mathbf{R}^1$. Then it follows that $\forall\psi \in \mathcal{H}$ and $\forall s \in \mathbf{R}^1$,

$$\left\langle \psi, e^{itH}e^{-isT}e^{-itH}\psi \right\rangle = \int_{\mathbf{R}^1} e^{-is\lambda}d \left\langle \psi, e^{itH}F(\lambda)e^{-itH}\psi \right\rangle = \int_{\mathbf{R}^1} e^{-is\lambda}d \left\langle \psi, F_t(\lambda)\psi \right\rangle$$

$$= \int_{\mathbf{R}^1} e^{-is(\lambda+t)}d \left\langle \psi, F(\lambda)\psi \right\rangle = \left\langle \psi, e^{-ist}e^{-isT}\psi \right\rangle.$$

By using the polarization identity, we can obtain the WR (8). According to von Neumann's uniqueness theorem, with respect to the solution of the WR [13], we had better to define T, which appears in the T-weak WR, as a symmetric operator, to allow the operator H (corresponding to the Hamiltonian) to be bounded from below. We note here that if a symmetric operator T satisfies the T-weak WR with some self-adjoint operator H, then the closure of T, denoted by $\overline{T}$, also satisfies the $\overline{T}$-weak WR with the same H. This is easily verified by usual calculation. It is guaranteed, from this proposition, that a symmetric operator T, satisfying the T-weak WR with the Hamiltonian for some system, is the time operator, and thus it is significant to examine the T-weak WR in the general analysis of the time operator. As a summary, we remark again that the following relations,

$$\text{WR} \ \Rightarrow \ T\text{-weak WR} \ \Rightarrow \ \text{CCR}$$

hold, in the sense of Proposition 2.1, even though, in general the converses do not hold, as is already mentioned in Sec. I.

III. THE AHARONOV-BOHM TIME OPERATOR

Let us consider the Hilbert space $L^2(\mathbf{R}^1)$. The operator T_0 on $L^2(\mathbf{R}^1)$ in Eq. (1),

$$T_0 := \frac{1}{2} \left(QP^{-1} \ + \ P^{-1}Q \right),$$

is defined in its domain $\text{Dom}(T_0) := \text{Dom}(QP^{-1}) \cap \text{Dom}(P^{-1}Q)$, where P is the momentum operator on $L^2(\mathbf{R}^1)$ for the 1DFPS, and P^{-1} its inverse. In the axiomatic quantum mechanics, P is defined as $P := -iD_x$, where D_x is a differential operator on $L^2(\mathbf{R}^1)$, and its domain consists of the L^2-functions which belong to $AC(\mathbf{R}^1)$, and satisfy that their

derivatives are also included in $L^2(\mathbf{R}^1)$ [5]. $AC(\Omega)$ (Ω is an open set of $\mathbf{R}^1$) is the set of functions on Ω, which are absolutely continuous on all bounded closed intervals of Ω. The free Hamiltonian H_0 for this system is $H_0 := P^2/2$. The position operator Q on $L^2(\mathbf{R}^1)$ is defined as an operator of multiplication by x on $L^2(\mathbf{R}^1)$, denoted by M_x, and its domain consists of L^2- functions, defined by ψ, such that $\int_{\mathbf{R}^1} |x\psi(x)|^2\, dx$ is finite. It is noted that in the definition of T_0, P^{-1} is well defined and becomes a self-adjoint operator on $L^2(\mathbf{R}^1)$. This is because, for any self-adjoint operator A, if its inverse A^{-1} exists, A^{-1} should be self-adjoint [15]. In our case, P^{-1} exists since P is an injection, i.e., $\mathrm{Ker}(P^{-1}) = \{0\}$, where $\mathrm{Ker}(A) := \{\psi \in \mathrm{Dom}(A) \mid A\psi = 0\}$.

In the momentum representation of T_0, we have

$$FT_0F^{-1} = \frac{1}{2}\left(iD_k M_{1/k} + M_{1/k}iD_k\right)$$

and its domain

$$\begin{aligned}
\mathrm{Dom}(FT_0F^{-1}) &= \mathrm{Dom}(D_k M_{1/k}) \cap \mathrm{Dom}(M_{1/k}D_k) \\
&= \left\{\psi \in \mathrm{Dom}(M_{1/k}) \mid M_{1/k}\psi \in \mathrm{Dom}(D_k)\right\} \\
&\quad \cap \left\{\psi \in \mathrm{Dom}(D_k) \mid D_k\psi \in \mathrm{Dom}(M_{1/k})\right\},
\end{aligned} \tag{9}$$

where F is the Fourier transformation from $L^2(\mathbf{R}^1)$ onto $L^2(\mathbf{R}^1_k)$, and the use has been made of the relations $FQF^{-1} = iD_k$, $FPF^{-1} = M_k$, and $FP^{-1}F^{-1} = M_{1/k}$. At first sight, $\mathrm{Dom}(T_0)$ seems to be rather restricted, because of the existence of P^{-1} in the definition of T_0. The following simple example by Kobe [1] may be considered to support this anticipation.

Example 1 : Let us consider the functions $\phi_n(k) := k^n N_n e^{-a_0 k^2} \in L^2(\mathbf{R}^1_k)$, where $n \in \mathbf{Z}$, $n \geq 0$, $a_0 > 0$ and N_n is a normalization factor. We see that for any integer $n \geq 2$, $\phi_n \in \mathrm{Dom}(FT_0F^{-1})$. The action of FT_0F^{-1} on each ϕ_n ($n \geq 2$) is, by direct calculation,

$$FT_0F^{-1}\phi_n(k) = \frac{i}{2}\left[(2n-1)k^{n-2} - 2a_0 k^{n-1} - 2a_0 k^n\right] N_n e^{-a_0 k^2} .$$

In the case of $n = 0$, 1, however, the right-hand side of the above equation is formally not square integrable, and thus ϕ_0, $\phi_1 \notin \mathrm{Dom}(FT_0F^{-1})$.

Notice that in spite of this example, $\mathrm{Dom}(T_0)$ is dense in $L^2(\mathbf{R}^1)$. This can be seen from the fact that the subspace $\mathcal{C}_i$ is included in $\mathrm{Dom}(FT_0F^{-1})$ and is dense in $L^2(\mathbf{R}^1_k)$. $\mathcal{C}_i$ is defined as

$$\mathcal{C}_i := \left\{\psi \in C_0^\infty(\mathbf{R}^1_k) \mid \mathrm{supp}\,\psi \subset \mathbf{R}^1_k \backslash \{0\}\right\}, \tag{10}$$

where $\mathrm{supp}\,\psi$ denotes the support of ψ, i.e. the closure of $\{k \in \mathbf{R}^1_k \mid \psi(k) \neq 0\}$. Therefore the adjoint operator of T_0, denoted by T_0^*, can be defined. Then, T_0 is symmetric, because

$$T_0^* \supset \frac{1}{2}\left((QP^{-1})^* + (P^{-1}Q)^*\right) \supset \frac{1}{2}\left((P^{-1})^*Q^* + Q^*(P^{-1})^*\right) = T_0,$$

where we have used the fact that $Q^* = Q$ and $(P^{-1})^* = P^{-1}$. It is noted that T_0 and H_0 do not satisfy the T_0-weak WR, $T_0 e^{-itH_0} = e^{-itH_0}(T_0 + t)$, $\forall t \in \mathbf{R}^1$. Because $\mathrm{Dom}(FT_0F^{-1})$ in Eq. (9) is not invariant under the action of $e^{-itM_{k^2}/2}$ for all $t \neq 0$, that is for any $t \neq 0$, there is some vector $\psi \in \mathrm{Dom}(FT_0F^{-1})$ satisfying $e^{-itM_{k^2}/2}\psi \notin \mathrm{Dom}(FT_0F^{-1})$. For instance, consider the following L^2-function g of $k \in \mathbf{R}^1_k$,

$$g(k) := \begin{cases} e^{-1/k^2}\dfrac{1}{1+|k|^s} & (k \neq 0) \\ 0 & (k = 0) \end{cases}$$

where $1/2 < s \leq 3/2$. Then g is C^∞-function. One can see that $g \in \mathrm{Dom}(FT_0F^{-1})$, however, $e^{-itM_{k^2/2}}g \notin \mathrm{Dom}(FT_0F^{-1})$, $\forall t \neq 0$. This follows from the fact that $e^{-itM_{k^2/2}}g \notin \mathrm{Dom}(D_k)$, $\forall t \neq 0$. We here introduce a symmetric extension of T_0 on $L^2(\mathbf{R}^1)$, denoted by $\widetilde{T_0}$, which will satisfy the $\widetilde{T_0}$-weak WR with H_0, and is defined, in the momentum representation, as follows,

$$
\mathrm{Dom}(F\widetilde{T_0}F^{-1}) := \left\{ \psi \in L^2(\mathbf{R}_k^1) \;\middle|\; \begin{array}{l} \psi \in AC(\mathbf{R}_k^1\backslash\{0\}), \;\; \lim_{k\to 0} \dfrac{\psi(k)}{|k|^{1/2}} = 0, \;\text{and} \\[2mm] \displaystyle\int_{\mathbf{R}_k^1\backslash\{0\}} \left| \dfrac{d\psi(k)/k}{dk} + \dfrac{1}{k}\dfrac{d\psi(k)}{dk} \right|^2 dk < \infty \end{array} \right\},
$$

and its action,

$$
F\widetilde{T_0}F^{-1}\psi(k) = \frac{i}{2}\left(\frac{d\psi(k)/k}{dk} + \frac{1}{k}\frac{d\psi(k)}{dk} \right), \quad \text{a.e. } k \in \mathbf{R}_k^1\backslash\{0\}, \;\; \forall\psi \in \mathrm{Dom}(F\widetilde{T_0}F^{-1}). \quad (11)
$$

It is seen that $\mathrm{Dom}(F\widetilde{T_0}F^{-1})$ is a subspace of $L^2(\mathbf{R}_k^1)$, and $F\widetilde{T_0}F^{-1}$ is a linear operator on $L^2(\mathbf{R}_k^1)$.

*Proposition **3.1*** : $\widetilde{T_0}$ *is a symmetric extension of* T_0.

Proof : $F\widetilde{T_0}F^{-1}$ being symmetric follows from that $\forall\psi, \forall\phi \in \mathrm{Dom}(F\widetilde{T_0}F^{-1})$,

$$
\int_{(0,\infty)} \bar{\phi}(k)\frac{i}{2}\left(\frac{d\psi(k)/k}{dk} + \frac{1}{k}\frac{d\psi(k)}{dk} \right)dk - \int_{(0,\infty)} \frac{-i}{2}\left(\frac{d\bar{\phi}(k)/k}{dk} + \frac{1}{k}\frac{d\bar{\phi}(k)}{dk} \right)\psi(k)dk
$$
$$
= i\left(\lim_{b\to\infty} \frac{\bar{\phi}(b)\psi(b)}{b} - \lim_{a\downarrow 0} \frac{\bar{\phi}(a)\psi(a)}{a} \right) = 0,
$$

where $\lim_{b\to\infty} \bar{\phi}(b)\psi(b)/b = 0$ and $\lim_{a\downarrow 0} \bar{\phi}(a)\psi(a)/a = 0$ are used. The former is brought from the integrability of $\bar{\phi}(k)\psi(k)$, and the latter from the boundary conditions of $\bar{\phi}(k)$ and $\psi(k)$ at the origin. By considering the left half-line in the same manner, we can obtain that $\forall\psi, \forall\phi \in \mathrm{Dom}(F\widetilde{T_0}F^{-1})$, $\left\langle \phi, F\widetilde{T_0}F^{-1}\psi \right\rangle = \left\langle F\widetilde{T_0}F^{-1}\phi, \psi \right\rangle$, that is, $F\widetilde{T_0}F^{-1}$ is symmetric. To see that $\widetilde{T_0}$ is an extension of T_0, i.e. $\widetilde{T_0} \supset T_0$, it is sufficient that every $\psi \in \mathrm{Dom}(FT_0F^{-1})$ satisfies the boundary condition at the origin, which appears in the definition of $\mathrm{Dom}(F\widetilde{T_0}F^{-1})$. This is easily verified as follows. Consider a $\psi \in \mathrm{Dom}(FT_0F^{-1})$ in Eq. (9) , then $\psi(k)/k$ belongs to $AC(\mathbf{R}_k^1)$, and thus, $\lim_{k\to 0} \psi(k)/k$ exists. Thus $\lim_{k\to 0} |\psi(k)|/|k|^{1/2}| = \lim_{k\to 0} |k|^{1/2}|\psi(k)/k| = 0$. $\qquad\square$

This operator may be more understood, from the view of the energy representation which was emphasized by Egusquiza and Muga, and many other authors (see [4] and the references therein).

It is, now, noted that $\mathrm{Dom}(\widetilde{T_0})$ is an invariant subspace of e^{-itH_0}. Because, for every $\psi \in \mathrm{Dom}(F\widetilde{T_0}F^{-1})$ and $t \in \mathbf{C}$ ($\mathrm{Im}\, t \leq 0$), $\lim_{k\to 0} e^{-itk^2/2}\psi(k)/|k|^{1/2} = 0$, and for almost everywhere $k \in \mathbf{R}_k^1\backslash\{0\}$,

$$
\frac{i}{2}\left(\frac{de^{-itk^2/2}\psi(k)/k}{dk} + \frac{1}{k}\frac{de^{-itk^2/2}\psi(k)}{dk} \right) = te^{-itk^2/2}\psi(k) + e^{-itk^2/2}\frac{i}{2}\left(\frac{d\psi(k)/k}{dk} + \frac{1}{k}\frac{d\psi(k)}{dk} \right),
$$

where the right-hand side is square-integrable. Therefore $e^{-itM_{k^2/2}}\psi$ is included in $\mathrm{Dom}(F\widetilde{T_0}F^{-1})$, and, as a result, $\widetilde{T_0}$ can satisfy the $\widetilde{T_0}$-weak WR with H_0,

$$
\widetilde{T_0}e^{-itH_0} = e^{-itH_0}(\widetilde{T_0} + t), \quad \forall t \in \mathbf{C} \;(\mathrm{Im}\, t \leq 0), \tag{12}
$$

whereas T_0 does not satisfy the T_0-weak WR with H_0. It is seen that $\widetilde{T_0}$ is not self-adjoint. Because, if it was so, $\widetilde{T_0}$ and H_0 would have to satisfy the WR from Proposition 2.1. The latter is however in contradiction to the nonnegativity $H_0 \geq 0$.

Consider the subspace $\mathcal{C}_i$ in Eq. (10). It is easily seen that $\mathcal{C}_i$ is an invariant subspace of $F\widetilde{T_0}F^{-1}$, that is, $F\widetilde{T_0}F^{-1} : \mathcal{C}_i \to \mathcal{C}_i$. Thus $F\widetilde{T_0}F^{-1}$ can act any times on $\mathcal{C}_i$. In the position representation, this property is described as $\widetilde{T_0} : F^{-1}\mathcal{C}_i \to F^{-1}\mathcal{C}_i$ where $F^{-1}\mathcal{C}_i := \{\psi \in L^2(\mathbf{R}^1) \mid \psi = F^{-1}\eta, \ \eta \in \mathcal{C}_i\}$. $\mathcal{C}_i$ may be regarded as an important subspace which determines the property of $\widetilde{T_0}$. Indeed, using the $\widetilde{T_0}$-weak WR in Eq. (12), we can obtain the following statement:

*Proposition **3.2*** : *For any nonnegative integer n, m and for any $\psi, \phi \in \mathcal{C}_i$,*

$$\lim_{t \to \pm\infty} |t|^n \left| \frac{d^m \left\langle \phi, e^{-itH_0}\psi \right\rangle}{dt^m} \right| = 0 \ .$$

That is, the probability amplitude, $\left\langle \phi, e^{-itH_0}\psi \right\rangle$, is a rapidly decreasing function of $t \in \mathbf{R}^1$.

Proof : Let $\psi, \phi \in F^{-1}\mathcal{C}_i$. Since $F^{-1}\mathcal{C}_i$ is an invariant subspace of $\widetilde{T_0}$, thus $\widetilde{T_0}\phi, \widetilde{T_0}\psi \in F^{-1}\mathcal{C}_i$. By using Eq. (12), we have

$$\left\langle \phi, e^{-itH_0}\widetilde{T_0}\psi \right\rangle = \left\langle \phi, (\widetilde{T_0} - t)e^{-itH_0}\psi \right\rangle = \left\langle \widetilde{T_0}\phi, e^{-itH_0}\psi \right\rangle - t\left\langle \phi, e^{-itH_0}\psi \right\rangle, \qquad (13)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbf{R}^1)$. Note that $\forall \psi \in L^2(\mathbf{R}^1)$, w-$\lim_{t \to \pm\infty} e^{-itH_0}\psi = 0$ [16], because H_0 is (spectrally) absolutely continuous [12]. This means that $\lim_{t \to \pm\infty} \left\langle \phi, e^{-itH_0}\widetilde{T_0}\psi \right\rangle = \lim_{t \to \pm\infty} \left\langle \widetilde{T_0}\phi, e^{-itH_0}\psi \right\rangle = 0$, which leads to the relation $\lim_{t \to \pm\infty} t\left\langle \phi, e^{-itH_0}\psi \right\rangle = 0$. In order to show that for any integer $n \geq 2$ and for any $\psi, \ \phi \in \mathcal{C}_i$, $\lim_{t \to \pm\infty} t^n \left\langle \phi, e^{-itH_0}\psi \right\rangle = 0$, we observe that $\widetilde{T_0}^k : F^{-1}\mathcal{C}_i \to F^{-1}\mathcal{C}_i$, and that the following relations similar to Eq. (13) hold for every $n \in \mathbf{N}$:

$$\left\langle \phi, e^{-itH_0}\widetilde{T_0}^{\,n+1}\psi \right\rangle = \left\langle \phi, (\widetilde{T_0} - t)^{\,n+1}e^{-itH_0}\psi \right\rangle$$

$$= \sum_{k=0}^{n}(-t)^k \binom{n+1}{k} \left\langle \widetilde{T_0}^{\,n+1-k}\phi, e^{-itH_0}\psi \right\rangle + (-t)^{n+1}\left\langle \phi, e^{-itH_0}\psi \right\rangle \ .$$

Then $\lim_{t \to \pm\infty} t^n \left\langle \phi, e^{-itH_0}\psi \right\rangle = 0$ is proved recursively for any integer $n \geq 2$. In order to show that $\left\langle \phi, e^{-itH_0}\psi \right\rangle$ is infinitely differentiable on $\mathbf{R}^1$, it is sufficient to use the fact that $\forall \psi \in F^{-1}\mathcal{C}_i$, $e^{-itH_0}\psi$ is infinitely and strongly differentiable on $\mathbf{R}^1$, and $F^{-1}\mathcal{C}_i$ is also an invariant subspace of H_0, that is, $H_0 : F^{-1}\mathcal{C}_i \to F^{-1}\mathcal{C}_i$. $\qquad \square$

We note that $\forall \psi, \phi \in \mathcal{C}_i$, $\left\langle \phi, e^{-itH_0}\psi \right\rangle$ converges to 0 as $t \to \pm\infty$, rapidly than any inverse-power of t. This fact is not trivial and is seen from the next example.

*Example **2*** : Define the survival probability of ψ as a function of $t \in \mathbf{R}^1$, i.e. $P_\psi(t) := \left| \left\langle \psi, e^{-itH_0}\psi \right\rangle \right|^2$, where ψ is an arbitrary element in $L^2(\mathbf{R}_k^1)$. Then, for a particular ϕ_n, $n \geq 2$ in Example 1, $P_{\phi_n}(t) = \left(1 + t^2/16a_0^2\right)^{-n-1/2}$ and this converges to 0 as $t \to \pm\infty$ as, at most, a power function of t.

From the above statement and example, we may expect that there is a connection between T_0 (or $\widetilde{T_0}$) and the survival probability. This expectation is also inspired from the works done by Bhattacharyya [17].

IV. CONNECTION BETWEEN THE TIME OPERATOR AND THE SURVIVAL PROBABILITY

If we assume the existence of a symmetric operator T which satisfies the T-weak WR with the Hamiltonian for some system, i.e. the time operator, several statements are derived, in a rigorous form, which concern to the connection between the time operator and the survival probability. Before deriving these statements, we introduce a few definitions. Let T be a symmetric operator on the Hilbert space $\mathcal{H}$ and define

$$\langle T \rangle_\psi := \langle \psi, T\psi \rangle, \quad (\Delta T)_\psi := \left\| \left(T - \langle T \rangle_\psi \right) \psi \right\|, \quad \forall \psi \in \mathrm{Dom}(T), \tag{14}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$, and $\| \cdot \|$ the norm in $\mathcal{H}$, defined by this inner product. $\langle T \rangle_\psi$ and $(\Delta T)_\psi$ are respectively called the expectation and standard deviation of T with respect to the state ψ.

Theorem 4.1 : *Let T be a symmetric operator on $\mathcal{H}$, and H be a self-adjoint operator on $\mathcal{H}$. Then if a pair of operators T and H satisfies the T-weak WR, the inequality (6) holds.*

Proof : Let us define self-adjoint operators $\cos(tH) := \left(e^{itH} + e^{-itH} \right)/2$ and $\sin(tH) := \left(e^{itH} - e^{-itH} \right)/2i$. Then, from the T-wea WR in Eq. (5), we can obtain two commutation relations

$$[T, \ \cos(tH)] = -it\sin(tH), \quad [T, \ \sin(tH)] = it\cos(tH). \tag{15}$$

From the above commutation relations, we can derive the uncertainty relations. From the first relation in Eq. (15), we have that

$$(\Delta T)_\psi^2 \, \| \cos(tH)\psi \|^2 \geq \frac{t^2}{4} \left| \langle \psi, [T, \cos(tH)] \, \psi \rangle \right|^2 = \frac{t^2}{4} \left| \mathrm{Im} \, \langle \psi, e^{-itH}\psi \rangle \right|^2,$$

$\forall \psi \in \mathrm{Dom}(T), \ \forall t \in \mathbf{R}^1$. Similarly, the second relation in Eq. (15) gives us an inequality

$$(\Delta T)_\psi^2 \, \| \sin(tH)\psi \|^2 \geq \frac{t^2}{4} \left| \mathrm{Re} \, \langle \psi, e^{-itH}\psi \rangle \right|^2, \quad \forall \psi \in \mathrm{Dom}(T), \ \forall t \in \mathbf{R}^1.$$

Adding these inequalities together, and taking into account of the relation $\| \cos(tH)\psi \|^2 + \| \sin(tH)\psi \|^2 = \| \psi \|^2$, the inequality (6) can be obtained. $\quad \square$

Two corollarys follow from the inequality (6).

*Corollary **4.2*** : *Let T be a symmetric operator on $\mathcal{H}$, and H be a self-adjoint operator on $\mathcal{H}$. Then if a pair of operators T and H satisfies the T-weak WR, T has no point spectrum.*

Proof : Suppose that there existed an eigenvector $\psi_0 \in \mathrm{Dom}(T)$ belonging to an eigenvalue $\lambda \in \mathbf{R}^1$ of T, that is, $T\psi_0 = \lambda\psi_0$ and $\|\psi_0\| = 1$. Then we see that $(\Delta T)_{\psi_0} = 0$, from the definition in Eq. (14). It follows, from Theorem 4.1, that $\langle \psi_0, e^{-itH}\psi_0 \rangle = 0, \forall t \in \mathbf{R}^1 \backslash \{0\}$. Since e^{-itH} is strongly continuous at any $t \in \mathbf{R}^1$, we have that $\|\psi_0\|^2 = \lim_{t \to 0} \langle \psi_0, e^{-itH}\psi_0 \rangle = 0$, and this is in contradiction to the premise. Thus T has no point spectrum. $\quad \square$

*Corollary **4.3*** : *Let T be a symmetric operator on $\mathcal{H}$, and H be a self-adjoint operator on $\mathcal{H}$. If a pair of operators T and H satisfies the T-weak WR, then H has no point spectrum.*

Proof : Since $\mathrm{Dom}(T)$ is dense in $\mathcal{H}$, for each $\psi \in \mathcal{H}$ there is a sequence $\{\psi_n\}_{n=1}^\infty \subset \mathrm{Dom}(T)$, satisfying $\psi_n \to \psi, \ n \to \infty$. It follows that

$$\left| \langle \psi, e^{-itH} \psi \rangle - \langle \psi_n, e^{-itH} \psi_n \rangle \right|$$
$$= \left| \langle \psi, e^{-itH} \psi \rangle - \langle \psi, e^{-itH} \psi_n \rangle + \langle \psi, e^{-itH} \psi_n \rangle - \langle \psi_n, e^{-itH} \psi_n \rangle \right|$$
$$\le \| e^{itH} \psi \| \| \psi - \psi_n \| + \| \psi - \psi_n \| \| e^{-itH} \psi_n \|$$
$$\le (\| \psi \| + \| \psi_n \|) \| \psi - \psi_n \|,$$

and thus,

$$\limsup_{t \to \pm\infty} \left| \langle \psi, e^{-itH} \psi \rangle \right| \le \limsup_{t \to \pm\infty} \left| \langle \psi_n, e^{-itH} \psi_n \rangle \right| + (\| \psi \| + \| \psi_n \|) \| \psi - \psi_n \|$$
$$= (\| \psi \| + \| \psi_n \|) \| \psi - \psi_n \|,$$

where we use the inequality (6) and $\psi_n \in \mathrm{Dom}(T)$, in the last equality. Note that above inequality holds for any $n \in \mathbf{N}$. Thus, in the limit $n \to \infty$, we obtain that

$$\forall \psi \in \mathcal{H}, \quad \lim_{t \to \pm\infty} \langle \psi, e^{-itH} \psi \rangle = 0. \tag{16}$$

This means that H has no point spectrum. Because if H has non-empty point spectrum, say $\lambda \in \mathbf{R}^1$, then there is a corresponding eigenvector ψ_λ, which satisfies $H\psi_\lambda = \lambda \psi_\lambda$. Obviously ψ_λ does not satisfy the above condition (16). □

Moreover, it is seen that H is absolutely continuous, under the same assumption as in Corollary 4.3. Its proof is, essentially, based on the theorem in [18]. For later convenience, we here introduce the closed subspace of $\mathcal{H}$, with respect to a self-adjoint operator H on $\mathcal{H}$, that is, $\mathcal{H}_{\mathrm{ac}}(H) := \{ \psi \in \mathcal{H} \mid \| E(\cdot) \psi \|^2 \text{ is absolutely continuous } \}$, where $\{ E(B) \mid B \in \mathbf{B}^1 \}$ is the spectral measure of H [19].

Theorem 4.4 : *Let T be a symmetric operator on $\mathcal{H}$, and H be a self-adjoint operator on $\mathcal{H}$. If a pair of operators T and H satisfies the T-weak WR, then*

$$\| E(B) \psi \|^2 \le \| T\psi \| \| \psi \| |B| \tag{17}$$

for all $\psi \in \mathrm{Dom}(T)$ and all $B \in \mathbf{B}^1$, where $|B|$ is the Lebesgue measure of B. In particular, H is absolutely continuous.

Proof : Let us, first, derive the inequality that, $\forall \epsilon > 0, \quad \forall \lambda \in \mathbf{R}^1, \quad$ and $\forall \psi \in \mathrm{Dom}(T)$,

$$\left| \mathrm{Im} \langle \psi, R(\lambda + i\epsilon) \psi \rangle \right| \le \pi \| T\psi \| \| \psi \|, \tag{18}$$

where $R(\lambda \pm i\epsilon) := (H - (\lambda \pm i\epsilon))^{-1}$. It is seen that

$$i \, \mathrm{Im} \langle \psi, R(\lambda + i\epsilon) \psi \rangle = \frac{1}{2} \int_{\mathbf{R}^1} \left(\frac{1}{\lambda' - \lambda - i\epsilon} - \frac{1}{\lambda' - \lambda + i\epsilon} \right) d \langle \psi, E(\lambda') \psi \rangle$$
$$= \frac{i}{2} \int_{\mathbf{R}^1} \left[\int_0^\infty (e^{-it(\lambda' - \lambda - i\epsilon)} + e^{it(\lambda' - \lambda + i\epsilon)}) dt \right] d \langle \psi, E(\lambda') \psi \rangle$$
$$= i \int_0^\infty e^{-\epsilon t} \langle \psi, \cos t(H - \lambda) \psi \rangle \, dt$$
$$= \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{e^{-\epsilon t}}{t} \langle \psi, [T, \sin t(H - \lambda)] \psi \rangle \, dt, \tag{19}$$

where, Fubini's theorem has been used in the third equality, and Eq. (15) in the last. To evaluate Eq. (19), it is sufficient to see that

$$\lim_{\delta \downarrow 0} \int_\delta^\infty \frac{e^{-\epsilon t}}{t} \langle T\psi, \sin t(H - \lambda) \psi \rangle \, dt. \tag{20}$$

We define, here, a function $f(\epsilon, \lambda) : (0, \infty) \times \mathbf{R}^1 \to \mathbf{R}^1$, as follows,

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$$f(\epsilon, \lambda) := \int_0^\infty e^{-\epsilon t} \frac{\sin t\lambda}{t} dt.$$

$f(\epsilon, \lambda)$ is continuous on $(0, \infty) \times \mathbf{R}^1$, because of the fact that $|e^{-\epsilon t} \sin t\lambda/t| \leq e^{-\epsilon t}|\lambda|$ for any $t > 0$, and of the use of the dominated convergence theorem. Furthermore, since $\forall \epsilon > 0$, $\forall \lambda \in \mathbf{R}^1$ $e^{-\epsilon t} \sin t\lambda$ is integrable on $[0, \infty)$, $f(\epsilon, \lambda)$ is differentiable with respect to any $\epsilon > 0$, for each fixed λ. Thus, it is obtained, through the partial integrations, that $\forall \lambda \neq 0$,

$$\partial_\epsilon f(\epsilon, \lambda) = -\frac{1}{\lambda} \frac{1}{1 + \epsilon^2/\lambda^2}.$$

Note that $\forall \lambda \in \mathbf{R}^1$, $\lim_{\epsilon \to \infty} f(\epsilon, \lambda) = 0$, we obtain

$$f(\epsilon, \lambda) = \pm \frac{\pi}{2} - \frac{1}{\lambda} \int_0^\epsilon \frac{1}{1 + \tau^2/\lambda^2} d\tau, \tag{21}$$

where each $\pm$ corresponds to the sign of λ. From this expression, $f(\epsilon, \lambda)$ is bounded, i. e. $|f(\epsilon, \lambda)| \leq \pi/2$. Eq. (20) is expressed by $f(\epsilon, \lambda)$,

$$\begin{aligned} \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{e^{-\epsilon t}}{t} \langle T\psi, \sin t(H - \lambda)\psi \rangle dt &= \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{e^{-\epsilon t}}{t} \left[\int_{\mathbf{R}^1} \sin t(\lambda' - \lambda) d \langle T\psi, E(\lambda')\psi \rangle \right] dt \\ &= \lim_{\delta \downarrow 0} \int_{\mathbf{R}^1} \left[\int_\delta^\infty e^{-\epsilon t} \frac{\sin t(\lambda' - \lambda)}{t} dt \right] d \langle T\psi, E(\lambda')\psi \rangle \\ &= \int_{\mathbf{R}^1} f(\epsilon, \lambda' - \lambda) d \langle T\psi, E(\lambda')\psi \rangle \\ &= \langle T\psi, f(\epsilon, H - \lambda)\psi \rangle, \end{aligned}$$

where Fubini's theorem is used in the second equality, and the dominated convergence theorem is in the third. Substituting above relation into Eq. (19),

$$i \operatorname{Im} \langle \psi, R(\lambda + i\epsilon)\psi \rangle = \langle T\psi, f(\epsilon, H - \lambda)\psi \rangle - \langle f(\epsilon, H - \lambda)\psi, T\psi \rangle.$$

Note that $\|f(\epsilon, H - \lambda)\| \leq \pi/2$, then Eq. (18) is obtained. Eq. (17) follows from Eq. (18) through Stone's formula. By virtues of Eq. (17) and the denseness of $\operatorname{Dom}(T)$ in $\mathcal{H}$, it is seen that H is absolutely continuous. □

V. ABSENCE OF MINIMUM-UNCERTAINTY STATES

When a pair of operators T and H satisfies the T-weak WR, the following uncertainty relation between them

$$(\Delta T)_\psi (\Delta H)_\psi \geq \frac{1}{2}, \qquad \forall \psi \in \operatorname{Dom}(TH) \cap \operatorname{Dom}(HT) \; (\|\psi\| = 1) \tag{22}$$

is automatically derived, from the CCR between T and H, the validity of which follows from Proposition 2.1 (a more detailed explanation will be given in the proof of Theorem 5.1). For operators Q and P in Sec. III, it is well known that there is a state $\psi \in \operatorname{Dom}(QP) \cap \operatorname{Dom}(PQ)$ ($\|\psi\| = 1$), which minimizes the uncertainty, that is, a Gaussian packet. The following statement, on the contrary, shows that, under some additional conditions, there is no state $\psi \in \operatorname{Dom}(TH) \cap \operatorname{Dom}(HT)$ ($\|\psi\| = 1$) which satisfies the equality in Eq. (22).

Theorem 5.1 : *Let T be a symmetric operator on $\mathcal{H}$, H be a self-adjoint operator on $\mathcal{H}$, and these operators satisfy the T-weak WR. Then if H is non negative and if the T-weak WR is analytically continued for all $t \in \mathbf{C}$ ($\operatorname{Im} t \leq 0$), the equality in Eq. (22) can never be satisfied by any $\psi \in \operatorname{Dom}(TH) \cap \operatorname{Dom}(HT)$ ($\|\psi\| = 1$).*

In order to prove this theorem, let us first consider two lemmas.

Lemma **5.2** : *Let T and H be symmetric operators on $\mathcal{H}$, and they satisfy the CCR $TH - HT = i$, on a subspace of $\mathrm{Dom}(TH) \cap \mathrm{Dom}(HT)$, denoted by $\mathcal{D}$. Then neither eigenvector of T nor that of H belongs to $\mathcal{D}$.*

Proof : Assume that an eigenvector of T, $\psi_\lambda \neq 0$, belonging to an eigenvalue λ exists in $\mathcal{D}$. Then $T\psi_\lambda = \lambda\psi_\lambda$, and thus we have $\langle \psi_\lambda, (TH - HT)\psi_\lambda \rangle = 0$. On the other hand, the condition in this lemma requires that $\langle \psi_\lambda, (TH - HT)\psi_\lambda \rangle = i\|\psi_\lambda\|^2 \neq 0$. Thus the subspace $\mathcal{D}$ contains no eigenvector of T. The rest of the proof for H can be done, as in the same way for T. □

Lemma **5.3** : *Let T and H be symmetric operators on $\mathcal{H}$, and them satisfy the CCR $TH - HT = i$, on a subspace of $\mathrm{Dom}(TH) \cap \mathrm{Dom}(HT)$, denoted by $\mathcal{D}$. If a state $\eta \in \mathcal{D}$ $(\|\eta\| = 1)$ and a pair of complex numbers $a, b \in \mathbf{C}$, satisfying the following two equalities*

$$(T + aH + b)\eta = 0, \tag{23}$$
$$\langle T\eta, H\eta \rangle + \langle H\eta, T\eta \rangle - 2 \langle T \rangle_\eta \langle H \rangle_\eta = 0, \tag{24}$$

exist, then Re $a = 0$ *and* Im $a > 0$.

Proof : Let η be a state which satisfies the conditions in this lemma. Then we have that

$$\langle T\eta, H\eta \rangle + \langle H\eta, T\eta \rangle = \langle \eta, (HT + i)\eta \rangle + \langle H\eta, T\eta \rangle = i - 2a\|H\eta\|^2 - 2b \langle \eta, H\eta \rangle \ .$$

From Eq. (23), we also have that

$$2 \langle T \rangle_\eta \langle H \rangle_\eta = -2 \langle \eta, (aH + b)\eta \rangle \langle \eta, H\eta \rangle = -2a \langle \eta, H\eta \rangle^2 - 2b \langle \eta, H\eta \rangle .$$

Therefore the condition Eq. (24) leads us to the relation

$$i - 2a\|H\eta\|^2 = -2a \langle \eta, H\eta \rangle^2 . \tag{25}$$

Let us consider the real and imaginary parts of the above equality, separately. It follows, from the real part (Re a)$\|H\eta\|^2 = $ (Re a) $\langle \eta, H\eta \rangle^2$, that Re $a = 0$. This is because if Re $a \neq 0$, then $\|H\eta\|^2 - \langle \eta, H\eta \rangle^2 = 0$ and this means that η is an eigenvector of H, belonging to the eigenvalue $\langle \eta, H\eta \rangle$, in spite of the premise $\eta \in \mathcal{D}$ $(\|\eta\| = 1)$. This is in contradiction to Lemma 5.2. It is also seen, from the imaginary part, $1 - 2(\mathrm{Im}\ a)\|H\eta\|^2 = -2(\mathrm{Im}\ a) \langle \eta, H\eta \rangle^2$, that Im $a > 0$. □

Proof of Theorem 5.1 : Let $\psi \in \mathrm{Dom}(TH) \cap \mathrm{Dom}(HT)$ and $\|\psi\| = 1$. Since the T-weak WR holds for T and H, the CCR in Eq. (2) follows. Then the uncertainty relation between T and H, in Eq. (22), is derived as

$$\begin{aligned}
(\Delta T)_\psi (\Delta H)_\psi &= \|(T - \langle T \rangle_\psi)\psi\| \|(H - \langle H \rangle_\psi)\psi\| \\
&\geq \left| \left\langle (T - \langle T \rangle_\psi)\psi, (H - \langle H \rangle_\psi)\psi \right\rangle \right| \\
&\geq \left| \mathrm{Im} \left\langle (T - \langle T \rangle_\psi)\psi, (H - \langle H \rangle_\psi)\psi \right\rangle \right| \\
&= \frac{1}{2} |\langle T\psi, H\psi \rangle - \langle H\psi, T\psi \rangle| = \frac{1}{2} \ .
\end{aligned}$$

In the second line, which is nothing but the Cauchy-Schwarz inequality, the equality holds if and only if there exists a complex number $\alpha \in \mathbf{C}$, satisfying

$$(T - \langle T \rangle_\psi)\psi + \alpha(H - \langle H \rangle_\psi)\psi = 0.$$

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In the third line, the equality holds if and only if

$$\text{Re } \left\langle (T - \langle T \rangle_\psi)\psi, (H - \langle H \rangle_\psi)\psi \right\rangle = \langle T\psi, H\psi \rangle + \langle H\psi, T\psi \rangle - 2 \langle T \rangle_\psi \langle H \rangle_\psi = 0.$$

In order to show that no $\psi \in \text{Dom}(TH) \cap \text{Dom}(HT)$ ($\|\psi\| = 1$) can satisfy the equality in the uncertainty relation between T and H, in Eq. (22), it is sufficient to see that, above two conditions can not be satisfied simultaneously, for any $\psi \in \text{Dom}(TH) \cap \text{Dom}(HT)$ ($\|\psi\| = 1$) and for any $\alpha \in \mathbf{C}$. Observe that these conditions take just the same form as the two equalities (23) and (24) in Lemma 5.3.

Let us now assume that there exist such a state $\eta \in \text{Dom}(TH) \cap \text{Dom}(HT)$ ($\|\eta\| = 1$), and a pair of complex numbers $a, b \in \mathbf{C}$, that satisfy both of Eqs. (23) and (24), and derive a contradiction. Lemma 5.3 implies that the parameter a is pure imaginary and is expressed as $a = iq$, $q > 0$, to lead $T\eta + iqH\eta + b\eta = 0$. Then we must have that $-q \langle \eta, H\eta \rangle = \text{Im } b \leq 0$, because $\langle \eta, T\eta \rangle \in \mathbf{R}^1$ and $H \geq 0$, (see the conditions of Theorem 5.1). It is also noted that e^{-itH} is bounded and $e^{-itH} H \subset H e^{-itH}$, for all $t \in \mathbf{C}$ (Im $t \leq 0$). Then it follows that $T e^{-itH}\eta = e^{-itH}(T\eta + t\eta) = (-iqH - b + t)e^{-itH}\eta$. Since Im $b \leq 0$ and Im $t \leq 0$, we can put $t = b$ and obtain $T e^{-ibH}\eta = -iqH e^{-ibH}\eta$. It is here noted that $e^{-ibH}\eta \neq 0$, because e^{-ibH} is an injection and $\eta \neq 0$. This is seen from the following relations,

$$\|e^{-ibH}\eta\|^2 = \int_{[0,\infty)} |e^{-ib\lambda}|^2 d\|E(\lambda)\eta\|^2 \geq \int_{[0,N]} |e^{-ib\lambda}|^2 d\|E(\lambda)\eta\|^2$$

$$\geq \inf_{\lambda \in [0,N]} |e^{-ib\lambda}|^2 \int_{[0,N]} d\|E(\lambda)\eta\|^2 = e^{-2|\text{Im } b|N} \int_{[0,N]} d\|E(\lambda)\eta\|^2,$$

where $\{E(B) \mid B \in \mathbf{B}^1\}$ is the specral measure of H, N is an arbitrary natural number, and we have used the fact that the spectrum of H should be included in $[0,\infty)$, because $H \geq 0$. Providing that $e^{-ibH}\eta = 0$, one would obtain that $0 = \lim_{N \to \infty} \int_{[0,N]} d\|E(\lambda)\eta\|^2 = \|\eta\|^2$, which contradicts $\|\eta\| = 1$. By taking the inner products between $e^{-ibH}\eta$ and each side of $T e^{-ibH}\eta = -iqH e^{-ibH}\eta$, we have

$$\left\langle e^{-ibH}\eta, T e^{-ibH}\eta \right\rangle = -iq \left\langle e^{-ibH}\eta, H e^{-ibH}\eta \right\rangle.$$

Notice that the both sides of this equality have to vanish. Since $q > 0$ and $e^{-ibH}\eta \neq 0$, we have that $0 = \left\langle e^{-ibH}\eta, H e^{-ibH}\eta \right\rangle = \left\langle H^{1/2}e^{-ibH}\eta, H^{1/2}e^{-ibH}\eta \right\rangle = \|H^{1/2}e^{-ibH}\eta\|^2$, where $H^{1/2}$ is a self-adjoint operator, satisfying $H = H^{1/2}H^{1/2}$ and $H^{1/2} \geq 0$. Thus $H e^{-ibH}\eta = 0$. It is also seen that $e^{-ibH}\eta \in \text{Dom}(TH) \cap \text{Dom}(HT)$, because $\eta \in \text{Dom}(TH) \cap \text{Dom}(HT)$. These facts are in contradiction to Lemma 5.2. Therefore, Eqs. (23) and (24) in Lemma 5.3 can not be satisfied simultaneously. This means that the equality, in the uncertainty relation between T and H in Eq. (22), never holds under the condition of Theorem 5.1. $\square$

The question about minimum-uncertainty states is motivated by the following result by Kobe [1],

$$\lim_{n \to \infty} \left(\Delta \widetilde{T_0}\right)_{F^{-1}\phi_n} (\Delta H_0)_{F^{-1}\phi_n} = \frac{1}{2} \ ,$$

where ϕ_n ($n \geq 2$) is defined as in Example 1. Note however, that ϕ_n does not converge in L^2-norm, as $n \to \infty$. This Kobe's result implies the absence of minimum-uncertainty states. This result was also derived by Wigner [20] and Baute et al [21], in different ways from ours. It should be notified that the absence of minimum-uncertainty states expresses a crucial difference between the Weyl relation and the T-weak WR.

VI. CONSTRUCTION OF THE TIME OPERATORS FOR GENERAL QUANTUM SYSTEMS

We first summarize the several results so far obtained about the time operator T_0 in Eq. (1), or an its extension $\widetilde{T_0}$ in Eq. (11), for the 1DFPS.

Example **3** : $\widetilde{T_0}$ satisfies the $\widetilde{T_0}$-weak WR with H_0, as is seen in Eq. (12). Then we have the following properties about $\widetilde{T_0}$.
(i)   The inequality (6) between $\left(\Delta\widetilde{T_0}\right)_\psi$ and the survival probability of ψ holds for all $\psi \in \mathrm{Dom}(\widetilde{T_0})$ (Theorem 4.1).
(ii) $\widetilde{T_0}$ has no point spectrum (Corollary 4.2).
(iii) The inequality (17) holds for all $\psi \in \mathrm{Dom}(\widetilde{T_0})$ and all $B \in \mathbf{B}^1$ (Theorem 4.4).
(iV) The uncertainty relation (22) between $\widetilde{T_0}$ and H_0 holds on $\mathrm{Dom}(\widetilde{T_0}H_0) \cap \mathrm{Dom}(H_0\widetilde{T_0})$ (Proposition 2.1), although there exists no state in $\mathrm{Dom}(\widetilde{T_0}H_0) \cap \mathrm{Dom}(H_0\widetilde{T_0})$, which satisfies equality in the uncertainty relation between $\widetilde{T_0}$ and H_0 (Theorem 5.1).

It is seen, from the above (i), that the Gaussian packet $F\phi_0$ in Example 1 is not included in $\mathrm{Dom}(\widetilde{T_0})$. Because, for large $|t|$, the survival probability of the Gaussian packet for the 1DFPS decays with an inverse-power law $|t|^{-1}$, and this is in contradiction to the behavior of the survival probability predicted by the inequality (6). It is, however, noticed that this kind of estimation about the domain $\mathrm{Dom}(\widetilde{T_0})$ is valid for one dimensional case. Because, as the dimension becomes higher, the survival probability decays faster, in general than $|t|^{-2}$. We also obtain, from the inequality (6) for $\widetilde{T_0}$ and H_0, that

$$2\sqrt{2}\left(\Delta\widetilde{T_0}\right)_\psi \geq \tau_h(\psi), \tag{26}$$

where $\tau_h(\psi)$ is defined as $\tau_h(\psi) := \sup\left\{ t \geq 0 \;\middle|\; \left|\langle\psi, e^{-itH_0}\psi\rangle\right|^2 = 1/2 \right\}$, $\forall\psi \in \mathrm{Dom}(\widetilde{T_0})$ ($\|\psi\| = 1$). This relation is important, to give the direct connection between $\left(\Delta\widetilde{T_0}\right)_\psi$ and the measurable quantity $\tau_h(\psi)$, although we don't know whether $\widetilde{T_0}$ itself is an observable. Let us also consider the physical meaning of the above (iii). The following inequality is derived from Eq. (17),

$$\|E_{H_0}(B)\psi\|^2 \leq \left(\Delta\widetilde{T_0}\right)_\psi |B|$$

for all $\psi \in \mathrm{Dom}(\widetilde{T_0})$ ($\|\psi\| = 1$) and $B \in \mathbf{B}^1$, because of the fact that the $\widetilde{T_0}$-weak WR is not changed, with replacing $\widetilde{T_0}$ by $\widetilde{T_0} - \left\langle\widetilde{T_0}\right\rangle_\psi$. Note that $\|E_{H_0}(B)\psi\|^2$ is the probability which one finds a measured energy-value in the range B for the fixed ψ. Suppose that $\left(\Delta\widetilde{T_0}\right)_\psi$ is small, then the probability $\|E_{H_0}(B)\psi\|^2$ should be uniformly small for all $B \in \mathbf{B}^1$. This concludes that the probability distribution $\|E_{H_0}(\cdot)\psi\|^2$ has a broad deviation, for $\|E_{H_0}(\mathbf{R}^1)\psi\|^2 = 1$. Hence $\left(\Delta\widetilde{H_0}\right)_\psi$ must be large and this result is consistent with the uncertainty relation.

In order to see the existence of the other quantum systems, than the 1DFPS, for which a time operator exists, let us recall the results obtained by Putnam, in the theory of Schrödinger operators [22]. According to this theorem, if a potential $V(x)$ is a real-valued measurable function on $\mathbf{R}^1$ satisfying $0 \leq V(x) \leq \mathrm{const}$, a.e. and $V(x) \in L^1(\mathbf{R}^1)$, then H_0 and $H_1 := H_0 + V(x)$ defined on $L^2(\mathbf{R}^1)$ are absolutely continuous, and furthermore

the wave operators $U_{\pm} := \text{s-}\lim_{t\to\pm\infty} e^{itH_1} e^{-itH_0}$ exist and are unitary operators satisfying $H_1 = U_{\pm} H_0 U_{\pm}^*$. For our purpose, we first define the operators $T_{1,\pm} := U_{\pm} \widetilde{T_0} U_{\pm}^*$ on $L^2(\mathbf{R}^1)$, where $U_{\pm}$ are the wave operators defined in this Putnam's theorem. Then $T_{1,\pm}$ are symmetric and satisfy the $T_{1,\pm}$-weak WR with H_1, i.e.

$$T_{1,\pm} e^{-itH_1} = e^{-itH_1} (T_{1,\pm} + t),$$

which are nothing but the unitary transformations of Eq. (12). These operators $T_{1,\pm}$ are the time operators we have sought for other quantum systems than the 1DFPS, and they satisfy all the properties, described in Example 3, with this H_1.

For a quantum system which allows bound states, we can also construct a time operator satisfying the T-weak WR with the Hamiltonian H, by restricting it to act on the set of scattering states. The latters are usually identified with the subspace $\mathcal{H}_{\text{ac}}(H)$ of the Hilbert space $\mathcal{H}$ under consideration. Because, in this case, the wave operator (if exits) is not a unitary operator on $\mathcal{H}$ in general, that is, the range $\text{Ran}(U_{\pm})$ becomes a proper subspace of $\mathcal{H}$. In fact, according to Kuroda [23], if the potential $V(x)$ is a real-valued measurable function on $\mathbf{R}^1$ satisfying $V(x) \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, $n \le 3$, and the Hamiltonian H_1 on $L^2(\mathbf{R}^1)$ is defined as $H_1 := H_0 + V(x)$, then the wave operators $U_{\pm}$ exist and are complete, i.e. $\text{Ran}(U_{\pm}) = L^2_{\text{ac}}(H_1)$, where $L^2_{\text{ac}}(H_1)$ is a subspace in $L^2(\mathbf{R}^1)$, similarly defined as $\mathcal{H}_{\text{ac}}(H)$ just before Theorem 4.4 . As in the same way for Putnam's theorem, by the use of the wave operators $U_{\pm}$ defined for H_0 and H_1 in this Kuroda's theorem, we can define the operators $T_{1,\pm} := U_{\pm} \widetilde{T_0} U_{\pm}^*$ on $L^2_{\text{ac}}(H_1)$ and their domains, $\text{Dom}(T_{1,\pm}) := U_{\pm} \text{Dom}(\widetilde{T_0})$. Then they are symmetric operators on $L^2_{\text{ac}}(H_1)$, and satisfy

$$T_{1,\pm} e^{-itH_{1,ac}} = e^{-itH_{1,ac}} (T_{1,\pm} + t).$$

$H_{1,ac}$ is defined as $H_{1,ac} := H_1|_{L^2_{\text{ac}}(H_1)} = U_{\pm} H_0 U_{\pm}^*$, and is called the (spectrally) absolutely continuous part of H_1 [12]. By the unitary equivalence, Example 3 is also valid for the pair $T_{1,\pm}$ and $H_{1,\text{ac}}$. We can not, however, extend these $T_{1,\pm}$ to the densely defined symmetric operators on $L^2(\mathbf{R}^1)$, so that they satisfy the $T_{1,\pm}$-weak WR with H_1, when H_1 has a point spectrum, i.e. $L^2_{\text{pp}}(H_1) \ne \{0\}$. This is because such an extension contradicts Corollary 4.3 and Theorem 4.4.

VII. CONCLUDING REMARKS

Analyzing the T-weak Weyl relation (T-weak WR) in Eq. (4), and obtaining several statements about the time operator, we have seen that the Aharonov-Bohm time operator T_0 in Eq. (1) (or an its symmetric extension $\widetilde{T_0}$ in Eq. (11)) is characterized by the $\widetilde{T_0}$-weak WR in Eq. (12). We have, in particular, recognized the fact that the time operator is deeply connected to the survival probability. In relation to this considerable connection, we would like, first, to revisit the inequality (6) in Theorem 4.1, Theorem 4.4, and their implications.

The inequality (6) is important to bring us a possibility of understanding the time operator from the two different points. The first point is related to the measurement of the survival probability. Since the inequality (26) derived from the inequality (6) gives the quantitative relation between the standard deviation of the time operator $\widetilde{T_0}$ and the maximum half-time of the survival probability in the 1DFPS, we may associate the time operator in quantum systems, with both the real and theoretical measurements of the survival probability. Another point is related to the connection with the dynamics of quantum systems. In order to see this possibility, we may refer to Proposition 3.2, and Corollary 4.3 which is one of the applications of the inequality (6). These facts imply a possibility of associating the time operator (or its domain), with the scattering state and its dynamics, through the survival probability.

As a remark on Theorem 4.4, the following suggestion by Putnam should be recalled, that is, the existence of the absolutely continuous part of the Hamiltonian can be inferred from

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the behavior of specific observables [24]. He considered the following system, in which there is a self-adjoint operator A_0 satisfying $A_t = A_0 + tI$, $\forall t \in \mathbf{R}^1$, where $A_t := e^{itH} A_0 e^{-itH}$ and H is the Hamiltonian for this system. He showed that H must be absolutely continuous (note that this is the case to which the last statement in Proposition 2.1 is applicable). The essence of its proof, that is, the uniqueness of the spectral resolution of A_0, also implies that if A_0 is maximally symmetric (not necessarily self-adjoint), H must be absolutely continuous. Because A_0 is uniquely represented by the generalized resolution of identity [25]. In this context, Theorem 4.4 is a generalization of the above statement by Putnam, to non-maximally symmetric operators.

The proof of the absolute continuity of H, which satisfies the T-weak Weyl relation with T, depends on Eq. (17). It is similar to the following inequality which was derived by Putnam [26] and Kato [8], on the study of the commutator of the form, $[H, iA] = C$, where H, A and C are bounded self-adjoint operators, and $C \geq 0$,

$$\|C^{1/2} E_H(B)\psi\|^2 \leq \|A\| \|\psi\|^2 |B|, \qquad \forall \psi \in \mathcal{H} \ \ \forall B \in \mathbf{B}^1. \tag{27}$$

It is seen that Eq. (17) is an unbounded case of theirs, and is not trivial. Because one can not replace directly C with the identity, for both bounded H and A. They showed, through Eq. (27), that H is absolutely continuous, provided that $\mathrm{Ker}(C) = \{0\}$, i.e. $\overline{\mathrm{Ran}(C)} = \mathcal{H}$. This statement is sometimes valid when the above H and A are unbounded, however, its proof originates in the more fundamental notion, T-smoothness [8], rather than the inequality like the above one, and the appropriate technique. Lavine applied it to Schrödinger operators in an especially good manner [9], [10], [11]. It should be noted that his work is closely related to the our problem. He found self-adjoint operators A which satisfy $[H, iA] = C$, with a certain class of Schrödinger operators H and positive bounded C. In the case of the absolute continuity of the free Hamiltonian H_0, we can choose the self-adjoint operator $A := (f(P)Q + Qf(P))/2$, where $f(P) := P(P^2 + \delta^2)^{-1}$ and $C := Pf(P)$. When $\delta > 0$, A is self-adjoint, because $\mathrm{Ker}(A^* \mp i) = \{0\}$. And also C is bounded and $C > 0$. They satisfy $[H_0, iA] = C$ and one can consider the case that a parameter δ approaches to 0, which is just $A = T_0$. Hence this scheme has the advance in the approach to the absolute continuity of H_0 and its connection with the Aharonov-Bohm time operator T_0.

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